



## Integrating load-balancing into multi-dimensional bin-packing problems

Trivella, Alessio; Pisinger, David

*Publication date:*  
2016

*Document Version*  
Peer reviewed version

[Link back to DTU Orbit](#)

*Citation (APA):*  
Trivella, A. (Author), & Pisinger, D. (Author). (2016). Integrating load-balancing into multi-dimensional bin-packing problems. Sound/Visual production (digital)

---

### General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

# Integrating load-balancing into multi-dimensional bin-packing problems

TECHNICAL UNIVERSITY OF DENMARK



ALESSIO TRIVELLA  
DAVID PISINGER



POZNAŃ, 06.07.2016

- 1 Recall the bin-packing problem
- 2 Integration of load-balancing: problem definition
- 3 MIP model for balancing a single bin
- 4 MIP model for joint packing and balancing
- 5 A multi-level local-search heuristic
- 6 Computational results

# Multi-dimensional Bin-Packing problem (MBP)

## Instance

- Set of  $D$ -dimensional rectangular-shaped boxes  
 $V = \{1, \dots, n\}$ . Box  $i$  has width =  $w_{i,d}$  in dimension  $d$
- Identical bins with width =  $W_d$  in dimension  $d$

## Problem

Orthogonally insert all boxes into the bins avoiding overlapping and using as few bins as possible. Rotations are not allowed



## Applications

Shipping and transportation industry, filling up containers, loading trucks etc. Most real-world problems have  $D \leq 3$ , but all results hold for any dimension

min  $N$

$$\text{s.t. } \sum_{d \in D} (l_{ijd} + l_{jid}) + p_{ij} + p_{ji} \geq 1 \quad \forall i < j \in V$$

$$x_{id} - x_{jd} + W_d l_{ijd} \leq W_d - w_{id} \quad \forall i \neq j \in V, d \in D$$

$$x_{id} \leq W_d - w_{id} \quad \forall i \in V, d \in D$$

$$a_i - a_j + n p_{ij} \leq n - 1 \quad \forall i \neq j \in V$$

$$1 \leq a_i \leq N \quad \forall i \in V$$

$$\text{var : } a_j, N \in \mathbb{N}, \quad x_{id} \in \mathbb{R}^+, \quad l_{ijd}, p_{ij} \in \{0, 1\} \quad \forall i, j \in V, d \in D$$

Difficult to solve in practice due to many:

- symmetries
- big-M constraints

# Load-Balanced MBP (LB-MBP)

## Instance

MBP instance + density of items  $\rho_i$  (or mass)

## Problem

Arrange items into the minimum number of bins, in such a way that the barycenters of the loaded bins fall as close as possible to an ideal point (e.g. the center of the bin or center of its base)

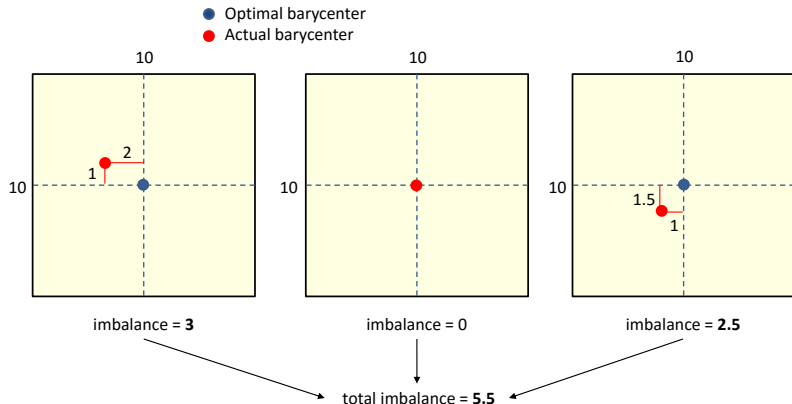
## Applications

Transport (ship, truck, aircraft's cargo): a good position of the center of mass increases the safety and efficiency of the travel, minimizing the waste of fuel

# Objective function

Minimize the total imbalance over:

- used bins
- dimensions



# Balancing a single bin

Assume to have a set  $V$  of items which fit into a single bin:

$$\begin{aligned} \min \quad & \sum_{d \in D} k_d (r_d + s_d) \\ \text{s.t. : } & r_d - s_d = B_d^{\text{opt}} - \frac{1}{M} \left( \sum_i m_i \left( x_{id} + \frac{w_{id}}{2} \right) \right) & \forall d \in D \\ & \sum_{d \in D} (l_{ijd} + l_{jid}) \geq 1 & \forall i < j \in V \\ & x_{id} - x_{jd} + W_d l_{ijd} \leq W_d - w_{id} & \forall i \neq j \in V, d \in D \\ & x_{id} \leq W_d - w_{id} & \forall i \in V, d \in D \\ \text{var : } & x_{id}, r_d, s_d \in \mathbb{R}^+ \quad l_{ijd} \in \{0, 1\} & \forall i, j \in V, d \in D \end{aligned}$$

To solve the LB-MBP we could:

- 1) Find the smallest number of bins
- 2) Balance each bin to optimality

...but the packing and balancing phases are not linked together!



# MIP model for the LB-MBP

$$\begin{aligned}
 \min \quad & NC + \sum_{d=1}^D \sum_{j=1}^N K_d (r_{jd} + s_{jd}) \\
 \text{s.t. : } \quad & \sum_{d=1}^D (l_{ijd} + l_{jid}) + p_{ij} + p_{ji} \geq 1 & \forall i < j \\
 & x_{id} - x_{jd} + W_d l_{ijd} \leq W_d - w_{id} & \forall i, j, \forall d \\
 & a_i - a_j + n p_{ij} \leq n - 1 & \forall i, j \\
 & x_{id} \leq W_d - w_{id} & \forall i \\
 & 1 \leq a_i \leq N & \forall i \\
 & n(c_{ij} - 1) \leq a_i - j \leq n(1 - c_{ij}) & \forall i, j \\
 & 1 - (n+1)(1 - \delta_{ij}) \leq a_i - j \leq -1 + (n+1)(1 - \gamma_{ij}) & \forall i, j \\
 & c_{ij} + \gamma_{ij} + \delta_{ij} = 1 & \forall i, j \\
 & m_i W_d (c_{ij} - 1) \leq e_{ijd} - m_i (x_{id} + w_{id}/2) \leq m_i W_d (1 - c_{ij}) & \forall i, j, \forall d \\
 & m_i W_d (c_{ij} - 1) \leq \alpha_{ijd} - m_i (W_d^{\text{opt}} - r_{jd} + s_{jd}) \leq m_i W_d (1 - c_{ij}) & \forall i, j, \forall d \\
 & e_{ijd} \leq c_{ij} W_d m_i & \forall i, j, \forall d \\
 & \alpha_{ijd} \leq c_{ij} W_d m_i & \forall i, j, \forall d \\
 & \sum_{i=1}^N e_{ijd} = \sum_{i=1}^N \alpha_{ijd} & \forall j, \forall d \\
 \text{var : } \quad & a_j, N \in \mathbb{N}, \quad x_{id}, r_{jd}, s_{jd}, e_{ijd}, \alpha_{ijd} \in \mathbb{R}^+, \quad l_{ijd}, p_{ij}, c_{ij}, \gamma_{ij}, \delta_{ij} \in \{0, 1\}
 \end{aligned}$$

# Sequential vs. joint problem

3D instance with 18 items,  $\rho_i = 1$ ,  $B^{opt} = (5, 5, 0)$

## Sequential problem

bin 1	1	2	3	4	5	7	8	10
bin 2	6	11						
bin 3	9	12	14	17				
bin 4	13	15	16	18				

## Joint problem

bin 1	1	2	3	5				
bin 2	4	6	7	8	10	12		
bin 3	9	11	14	17				
bin 4	13	15	16	18				

Optimal 3DBPP: uses 4 bins

bin	$B_x$	$B_y$	$B_z$	$f_{bin}$
bin 1	5.00	5.00	4.18	4.18
bin 2	6.12	5.00	4.44	5.56
bin 3	5.00	5.00	4.38	4.38
bin 4	5.00	5.00	3.32	3.32
$f_{coord}$	1.12	0.00	16.32	<b>17.44</b>

Different 3DBPP solution

bin	$B_x$	$B_y$	$B_z$	$f_{bin}$
bin 1	5.00	5.00	4.22	4.22
bin 2	5.00	5.00	3.64	3.64
bin 3	5.00	5.00	4.59	4.59
bin 4	5.00	5.00	3.32	3.32
$f_{coord}$	0.00	0.00	15.77	<b>15.77</b>

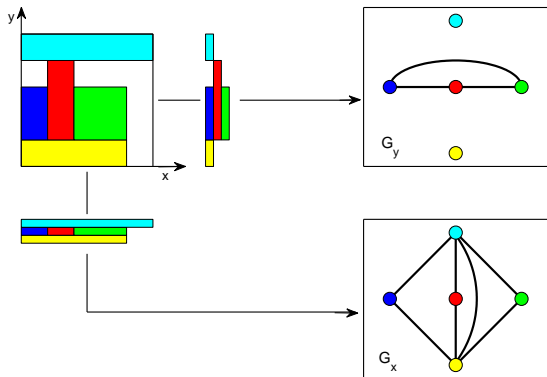
10% improvement!

But running time is 4 vs. 132 seconds. In general the joint model cannot be solved for instances larger than 15-20 items

# Heuristic load balancing

We now develop a heuristic algorithm to solve large instances

It is possible to characterize feasible packings by means of a set of **Interval Graphs** (Fekete-Schepers)



## Theorem 1

If  $D$  graphs  $G_d$ ,  $d \in D$ , are obtained from a packing, then the following conditions are fulfilled:

$P_1$  : *Each  $G_d$  is an interval graph*

$P_2$  :  $\cap_d G_d = \emptyset$

$P_3$  : *The stable sets of  $G_d$  have total weight less than the  $d$ -dimension of the bin*

## Definition

Let  $G$  be an undirected graph. An orientation  $\Phi$  of  $G$  is called **transitive orientation** (TRO) if:

$$(a, b) \in \Phi \wedge (b, c) \in \Phi \implies (a, c) \in \Phi$$

### Theorem 2

If  $G$  is an interval graph, then its complement  $\overline{G}$  is transitively orientable

### Theorem 3

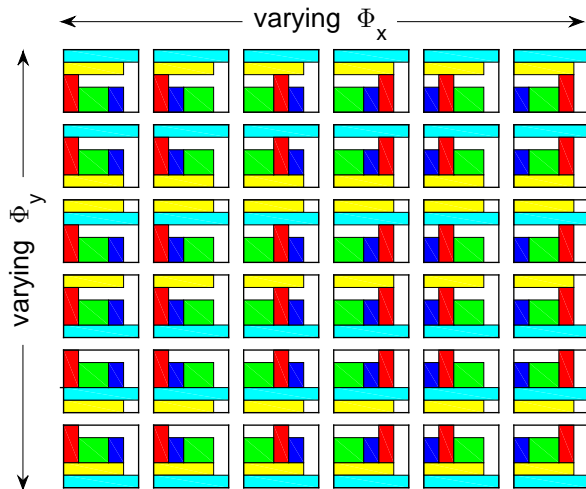
Let  $G_d$ ,  $d \in D$  be  $D$  graphs satisfying  $P_1$ ,  $P_2$ ,  $P_3$ , and call  $\Phi = (\Phi_d)_{d \in D}$ , where  $\Phi_d$  is a transitive orientation of  $\overline{G}_d$ .

The function  $p^\Phi : V \longrightarrow \mathbb{R}_0^{+D}$  defined by:

$$p_d^\Phi(v) = \begin{cases} 0 & \text{if } \nexists u \in V : (u, v) \in \Phi_d \\ \max\{p_d^\Phi(u) + w_d(u) \mid (u, v) \in \Phi_d\} & \text{otherwise} \end{cases}$$

produces a packing

# How many transitive orientations?



Different transitive orientations produce different packings

### How many TROs?

From graph theory: number of TROs of a graph is  $\prod_{i=1}^k r_i!$  where  $r_i$  is the number of vertices of particular substructures

### Is it possible to find TROs?

From graph theory: we can characterize them all TROs of a graph (it's complicated though)

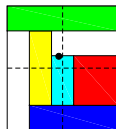
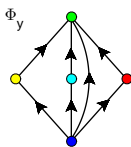
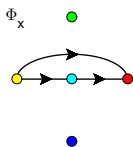
### Local Search

We define a best-improvement local search exploring a quadratic neighborhood of TROs. For each TRO:

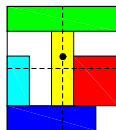
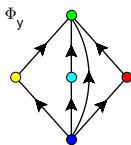
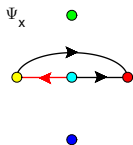
- go back to the corresponding packing
- evaluate the load balancing

# Example for a 2D case

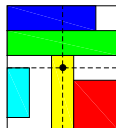
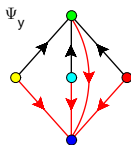
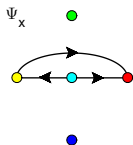
Items have different densities,  $\text{bin} = 5 \times 5$ ,  $B_{\text{opt}} = (2.5, 2.5)$



$B = (2.32, 2.97)$



$B = (2.50, 2.97)$



$B = (2.50, 2.50)$

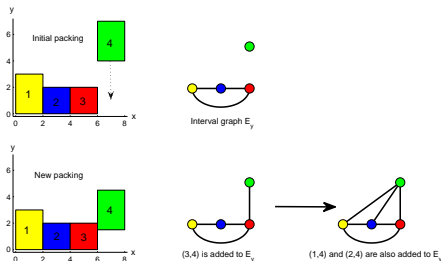


# Local search at graph level

**Purpose:** improve cases where the number of TROs is limited

**How:** modifying the structure itself of the graphs:

- Consider interval graphs  $G_d$
- Add or remove edges using specific rules (Crainic et al.)



- If new graphs correspond to a packing, then start local search on TROs

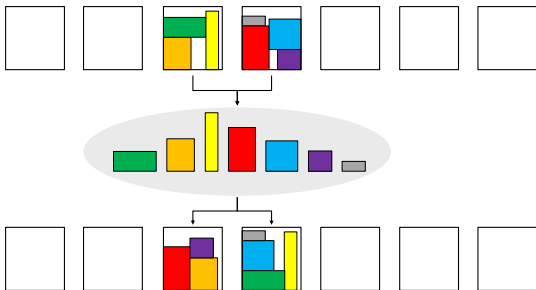
# Local search at bin-packing level (1/2)

## Purpose

Exploit the balancing potential of having a large number of bin-packings solutions with the same number of bins

## How

Iteratively repack and rebalance n-tuples of weakly balanced bins using *variable-depth neighborhood search* (VDNS)



Define a *k-neighborhood* as the set of all bin-packing solutions obtained by repacking at most  $k$  bins

### VDNS algorithm

- 1 Assign imbalance scores to the bins
- 2 Select  $k$  bins using *roulette wheel selection*
- 3 Repack the bins using a heuristic for MBP
- 4 If  $k$  bins are still used, balance them
- 5 If balancing is improved: save solution and update scores

$k$  is dynamically adjusted:

if no solutions are found after  $n$  iterations:  $k = k + 1$

# Results for 3D Bin-Packing instances (1/2)

Optimal barycenter  $\left(\frac{W_1}{2}, \frac{W_2}{2}, \frac{W_3}{2}\right)$ : center of the bin

cl.	size	LB	bins	$\rho_i = 1$			$\rho_i \sim U(1, 2)$			$\rho_i \sim U(1, 6)$		
				Init.	Bin	VDNS	Init.	Bin	VDNS	Init.	Bin	VDNS
1	100	24.12	25.64	16.48	0.91	<b>0.011</b>	17.03	0.68	<b>0.054</b>	19.22	2.26	<b>0.201</b>
2	100	24.64	26.12	16.19	0.85	<b>0.011</b>	16.69	0.68	<b>0.042</b>	19.01	2.28	<b>0.198</b>
3	100	24.48	26.08	16.34	0.87	<b>0.006</b>	16.78	0.66	<b>0.030</b>	19.16	2.21	<b>0.245</b>
4	100	57.44	60.60	31.74	0.58	<b>0.011</b>	30.64	0.29	<b>0.030</b>	30.91	0.93	<b>0.090</b>
5	100	13.60	14.60	15.83	0.92	<b>0.003</b>	15.87	0.71	<b>0.068</b>	17.65	2.70	<b>0.425</b>
6	100	18.20	20.08	9.83	1.45	<b>0.190</b>	10.30	1.70	<b>0.396</b>	13.49	4.54	<b>1.210</b>
7	100	11.12	12.36	16.89	1.19	<b>0.012</b>	16.53	0.79	<b>0.050</b>	18.37	3.11	<b>0.559</b>
8	100	15.52	17.08	15.66	0.78	<b>0.008</b>	15.34	0.64	<b>0.056</b>	17.35	2.63	<b>0.501</b>
1	200	48.84	51.16	15.05	0.79	<b>0.007</b>	15.74	0.82	<b>0.032</b>	18.36	2.57	<b>0.208</b>
2	200	48.48	50.80	14.81	0.77	<b>0.006</b>	15.57	0.85	<b>0.044</b>	18.24	2.58	<b>0.217</b>
3	200	49.24	51.24	14.91	0.77	<b>0.005</b>	15.68	0.83	<b>0.036</b>	18.29	2.65	<b>0.262</b>
4	200	117.8	122.2	31.91	0.53	<b>0.004</b>	30.66	0.29	<b>0.012</b>	30.91	0.96	<b>0.034</b>
5	200	25.60	27.36	12.47	0.75	<b>0.011</b>	13.06	0.89	<b>0.147</b>	14.90	3.23	<b>0.801</b>
6	200	35.84	38.24	6.74	1.30	<b>0.241</b>	8.17	2.19	<b>0.523</b>	12.09	5.44	<b>1.505</b>
7	200	20.36	22.40	12.72	1.26	<b>0.057</b>	12.95	1.15	<b>0.329</b>	14.89	3.99	<b>1.567</b>
8	200	29.40	32.04	12.75	0.70	<b>0.016</b>	12.79	0.98	<b>0.183</b>	14.66	3.33	<b>0.921</b>

- Results are average over 25 instances, running time is < 5-10s

# Results for 3D Bin-Packing instances (2/2)

Optimal barycenter  $\left(\frac{W_1}{2}, \frac{W_2}{2}, 0\right)$ : center of the base

cl.	size	LB	bins	$\rho_I = 1$				$\rho_I \sim U(1, 2)$				$\rho_I \sim U(1, 6)$			
				Init.	Bin	VDNS	LLB	Init.	Bin	VDNS	LLB	Init.	Bin	VDNS	LLB
1	100	24.12	25.64	52.30	43.75	<b>41.98</b>	36.06	51.57	42.53	<b>41.64</b>	31.87	54.78	41.93	<b>40.84</b>	26.82
2	100	24.64	26.12	61.88	46.52	<b>44.49</b>	35.90	58.89	43.95	<b>43.01</b>	31.72	59.32	40.21	<b>39.35</b>	26.68
3	100	24.48	26.08	52.64	43.58	<b>41.92</b>	35.56	52.00	42.49	<b>41.46</b>	31.42	53.08	41.56	<b>40.45</b>	26.41
4	100	57.44	60.60	60.92	39.72	<b>38.42</b>	25.59	58.72	38.57	<b>38.23</b>	22.61	58.82	38.08	<b>37.58</b>	19.06
5	100	13.60	14.60	54.65	44.83	<b>42.35</b>	36.77	51.78	42.70	<b>41.70</b>	32.44	51.97	41.48	<b>40.27</b>	27.16
6	100	18.20	20.08	53.07	47.26	<b>44.91</b>	41.98	51.33	45.63	<b>44.24</b>	37.07	51.92	44.49	<b>43.13</b>	31.13
7	100	11.12	12.36	55.24	44.94	<b>42.36</b>	36.59	51.87	42.58	<b>41.59</b>	32.30	51.96	41.80	<b>39.80</b>	27.12
8	100	15.52	17.08	54.77	44.83	<b>42.08</b>	37.08	51.97	43.03	<b>41.99</b>	32.72	52.37	42.05	<b>40.45</b>	27.44
1	200	48.84	51.16	51.98	44.00	<b>42.56</b>	36.87	51.77	43.18	<b>42.14</b>	32.58	52.90	42.31	<b>41.09</b>	27.43
2	200	48.48	50.80	61.86	47.10	<b>45.20</b>	36.79	59.34	44.72	<b>43.65</b>	32.51	59.96	40.72	<b>39.93</b>	27.36
3	200	49.24	51.24	52.38	44.11	<b>42.67</b>	36.68	52.18	43.26	<b>42.22</b>	32.14	53.45	42.35	<b>41.16</b>	27.28
4	200	117.8	122.2	60.31	39.50	<b>38.19</b>	25.74	58.29	38.40	<b>38.05</b>	22.76	58.55	37.92	<b>37.36</b>	19.18
5	200	25.60	27.36	53.89	45.90	<b>44.14</b>	39.29	52.23	44.57	<b>43.48</b>	34.71	52.93	43.54	<b>42.16</b>	29.21
6	200	35.84	38.24	52.24	48.28	<b>45.77</b>	44.36	51.06	46.74	<b>45.12</b>	39.19	51.83	45.34	<b>43.64</b>	32.97
7	200	20.36	22.40	53.96	46.47	<b>44.51</b>	39.88	52.31	44.87	<b>43.90</b>	35.23	53.43	44.04	<b>42.72</b>	29.64
8	200	29.40	32.04	53.68	45.84	<b>44.37</b>	39.61	52.14	44.73	<b>43.72</b>	34.98	53.09	45.84	<b>42.40</b>	29.42

- Results are average over 25 instances, running time is  $< 5$ -10s
- LLB is a lower bound obtained from “liquifying” the items

Alessio Trivella and David Pisinger. “The load-balanced multi-dimensional bin-packing problem”. *Computers & Operations Research* 74 (2016)152–164



Thank you.